Received: *February 09, 2001*, Accepted: *April 20, 2001*
Revised: *April 10, 2001*

Non-commutative string worldsheets from matrix models

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Abstract: We study dynamical effects of introducing noncommutativity on string worldsheets by using a matrix model obtained from the zero-volume limit of four-dimensional SU(N) Yang-Mills theory. Although the dimensionless noncommutativity parameter is of order $1/N$, its effect is found to be non-negligible even in the large-N limit due to the existence of higher Fourier modes. We find that the Poisson bracket grows much faster than the Moyal bracket as we increase N, which means in particular that the two quantities do not coincide in the large-N limit. The well-known instability of bosonic worldsheets due to long spikes is shown to be cured by the noncommutativity. The extrinsic geometry of the worldsheet is described by a crumpled surface with a large Hausdorff dimension.

Keywords: Bosonic Strings, Non-Commutative Geometry, Matrix Models.

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JHEP04(2001)024

Contents

1. Introduction

It is well known that the traditional bosonic worldsheet theories described for example by the Nambu-Goto action, the Polyakov action and the Schild action are not well-defined non-perturbatively, as was first observed in the large dimension limit by Alvarez [1]. This conclusion has been confirmed more directly by the dynamical triangulation approach [2, 3], where the discretized worldsheet embedded in the target space degenerates to long spikes, and hence one cannot view it as a proper approximation of a continuous worldsheet. It is therefore of interest to inquire whether it is possible to have different types of string models, where the worldsheet is well defined in the sense that it does not degenerate into long spikes. Some attempts have been made to make the string more "stiff" by the introduction of extrinsic curvature [4], but it is not clear whether this will work non-perturbatively [5].

In the present paper we have investigated whether the introduction of noncommutativity on the worldsheet changes the situation. We study such a system by using a matrix model with $N \times N$ hermitean matrices A_μ with $\mu = 1, 2, 3, 4$. Corresponding to each matrix one can construct a field $X_\mu(\sigma)$ by the Weyl transformation. Here σ stands for the two discrete variables σ_1 and σ_2, representing the worldsheet coordinates. Then there exists the following relation between the partition functions in the matrix model and in the corresponding Weyl-transformed model,

$$\int \mathrm{d}A_\mu \exp\left(\frac{1}{4g^2}\,\mathrm{Tr}[A_\mu, A_\nu]^2\right) = \int \mathcal{D}X_\mu(\sigma)\,\mathrm{e}^{-S}\,, \tag{1.1}$$

where the action S for the latter is given by

$$S = -\frac{1}{4g^2 N}\sum_\sigma \Big(X_\mu(\sigma)\star X_\nu(\sigma) - X_\nu(\sigma)\star X_\mu(\sigma)\Big)^2\,. \tag{1.2}$$

Here $\star$ denotes the star product with the dimensionless non-commutative parameter being of order $1/N$. The partition function on the l.h.s. of eq. (1.1) was, based on a combination of analytic and numerical arguments, first conjectured to exist in the large-N limit in ref. [6]. Monte Carlo simulations up to $N = 256$ [7] confirmed this conjecture.[1] This means that the corresponding Weyl-transformed model is also well defined in the large-N limit.

The partition function on the r.h.s. of eq. (1.1) defines a non-commutative two-dimensional field theory, which is invariant under the star-unitary transformation,

$$X_\mu(\sigma) \longrightarrow g^*(\sigma)\star X_\mu(\sigma)\star g(\sigma) \qquad \text{with} \qquad g^*(\sigma)\star g(\sigma) = g(\sigma)\star g^*(\sigma) = 1\,. \tag{1.3}$$

The action (1.2) resembles the Schild action

$$S_{\mathrm{Schild}} = \frac{1}{g^2 N}\int \mathrm{d}^2\sigma\left(\frac{\partial X_\mu}{\partial\sigma_1}\frac{\partial X_\nu}{\partial\sigma_2} - \frac{\partial X_\nu}{\partial\sigma_1}\frac{\partial X_\mu}{\partial\sigma_2}\right)^2, \tag{1.4}$$

the difference being that the Poisson bracket has been replaced by the Moyal bracket in the non-commutative case. It has been pointed out long time ago [9] that if the higher modes in the mode expansion of $X_\mu(\sigma)$ can be neglected, the star-Schild action in (1.2) reduces to the Schild action (1.4) in the large-N limit. The issue is also relevant when one considers matrix models as a non-perturbative definition of M-theory and type-IIB superstring theory [10, 11]. Of course, whether the higher modes can really be neglected or not is a non-trivial dynamical question. One of the main results of this paper is that for the bosonic model we consider, the Poisson and the Moyal brackets are very different due to the effect of the higher modes, and that this effect increases with increasing N. The agreement (or disagreement) of the two actions in the large-N limit is also discussed in ref. [12].

Recently the Schild action has been investigated non-perturbatively from the point of view of dynamical triangulation [3]. The result is similar to the one obtained for the Nambu-Goto action, namely that it is dynamically favorable to have a "surface" which degenerates to long spikes, and hence the notion of a worldsheet looses its meaning.[2]

[1] It has been proved rigorously in a preprint [8], which appeared while our paper was being revised.

[2] A similar result is not valid in the supersymmetric case, where the worldsheet exists if the fermions are coupled to a bosonic Schild action [3]. However, in the present paper we shall discuss the bosonic case only.

We shall take a different approach and ask whether the star-Schild action in (1.2) provides a bosonic string theory with a well-defined worldsheet. The effects of non-commutativity are found to be drastic. The average link length is finite and it is observed to be considerably smaller than the average extent of the surface, which is in sharp contrast [1, 3] to the conclusion for the Schild action (1.4) that the worldsheet becomes completely unstable due to long spikes. This is not a contradiction since the Poisson and Moyal brackets are indeed found to be quite different for the theory under consideration. On the other hand, the extrinsic geometry of the worldsheet for the star-Schild action is described by a crumpled surface with a large Hausdorff dimension.

The matrix model on the l.h.s. of eq. (1.1) can be regarded as the hermitean matrix version of the Eguchi-Kawai model [13]. (See refs. [14, 15] for the hermitean matrix version of the Eguchi-Kawai model with quenching [16] or twists [17], respectively.) Large-N scaling of SU(N) invariant correlation functions was obtained analytically to all orders of the $1/D$ expansion [7], and the results were reproduced by Monte Carlo simulations [7, 18]. In particular, the one-point function of a Wilson loop was observed to obey the area law, which suggested that the model is actually equivalent to large-N gauge theory for a finite region of scale even *without* quenching [16] or twists [17]. The non-commutative worldsheet theory studied in the present paper is therefore related to the large-N QCD string (For a recent review, see [19].).

The organization of this paper is as follows. In section 2 we briefly review the map from matrices to non-commutative fields. In section 3 we interpret the matrix model as non-commutative worldsheet theory. We then discuss the star-unitary invariance and the important question of gauge fixing. In section 4 the results are presented. section 5 contains the summary and discussions. In an appendix we make some comments on the relationship between the star-unitary invariance and the area-preserving diffeomorphisms.

2. From matrices to non-commutative fields

In order to derive the equivalence (1.1) between the matrix model and the non-commutative worldsheet theory, we briefly review the one-to-one correspondence between matrices and non-commutative fields. Most of the results in this section are known in the literature (see e.g., [20, 21]), but are given in order to fix the notation.

Throughout this paper, we assume that N is odd.[3] We first introduce the 't Hooft-Weyl algebra

$$\Gamma_1 \Gamma_2 = \omega\, \Gamma_2 \Gamma_1 \,, \tag{2.1}$$

[3]We expect that the large-N dynamics of the matrix model on the l.h.s. of (1.1) is independent of whether N is even or odd. This has been also checked numerically for various SU(N) invariant quantities. However, the one-to-one correspondence between the matrices and non-commutative fields works rigorously only for odd N.

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where $\omega = \mathrm{e}^{4\pi i/N}$. It is known that the representation of Γ_i using $N \times N$ unitary matrices is unique up to unitary transformation $\Gamma_i \to U\Gamma_i U^\dagger$, where $U \in \mathrm{SU}(N)$. An explicit representation can be given by

$$\Gamma_1 = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{(N-1)} \end{pmatrix}. \tag{2.2}$$

Then we construct $N \times N$ unitary matrices

$$J_k \stackrel{\mathrm{def}}{=} (\Gamma_1)^{k_1}(\Gamma_2)^{k_2}\ \mathrm{e}^{-2\pi i k_1 k_2/N}\,, \tag{2.3}$$

where k is a 2d integer vector and the phase factor $\mathrm{e}^{-2\pi i k_1 k_2/N}$ is included so that

$$J_{-k} = (J_k)^\dagger\,. \tag{2.4}$$

Since $(\Gamma_i)^N = 1$, the matrix J_k is periodic with respect to k_i with period N,

$$J_{k_1+N,k_2} = J_{k_1,k_2+N} = J_{k_1,k_2}\,. \tag{2.5}$$

We then define the $N \times N$ matrices

$$\Delta(\sigma) \stackrel{\mathrm{def}}{=} \sum_{k_i \in \mathbb{Z}_N} J_k\ \mathrm{e}^{2\pi i k\cdot\sigma/N}\,, \tag{2.6}$$

labelled by a 2d integer vector σ. Note that $\Delta(\sigma)$ is hermitean due to the property (2.4). It is periodic with respect to σ_i with period N,

$$\Delta(\sigma_1 + N, \sigma_2) = \Delta(\sigma_1, \sigma_2 + N) = \Delta(\sigma_1, \sigma_2)\,. \tag{2.7}$$

It is easy to check that $\Delta(\sigma)$ possesses the following properties.

$$\mathrm{tr}\,\Delta(\sigma) = N\,, \tag{2.8}$$

$$\sum_{\sigma_i \in \mathbb{Z}_N} \Delta(\sigma) = N^2 \mathbf{1}_N\,, \tag{2.9}$$

$$\frac{1}{N}\,\mathrm{tr}\Big(\Delta(\sigma)\Delta(\sigma')\Big) = N^2 \delta_{\sigma,\sigma'\ (\mathrm{mod}\ N)}\,. \tag{2.10}$$

Now let us consider gl(N,$\mathbb{C}$), the linear space of $N \times N$ complex matrices. The inner product can be defined by $\mathrm{tr}(M_1^\dagger M_2)$, where M_1,$M_2 \in$ gl(N,$\mathbb{C}$). Equation (2.10) implies that $\Delta(\sigma)$ are mutually orthogonal and hence linearly independent. Since the dimension of the linear space gl$(N, \mathbb{C})$ is N^2 and there are $N^2 \Delta(\sigma)$'s that are linearly independent, $\Delta(\sigma)$ actually forms an orthogonal basis of gl$(N, \mathbb{C})$. Namely, any $N \times N$ complex matrix M can be written as

$$M = \frac{1}{N^2} \sum_{\sigma_i \in \mathbb{Z}_N} f(\sigma)\Delta(\sigma)\,, \tag{2.11}$$

where $f(\sigma)$ is a complex-valued function on the 2d discretized torus obeying periodic boundary conditions. Using orthogonality (2.10), one can invert (2.11) as

$$f(\sigma) = \frac{1}{N}\,\mathrm{tr}\Big(M\,\Delta(\sigma)\Big)\,. \tag{2.12}$$

The fact that (2.11) and (2.12) hold for an arbitrary $N \times N$ complex matrix M implies that $\Delta(\sigma)$ satisfies

$$\frac{1}{N^2}\sum_{\sigma_i\in\mathbb{Z}_N}\Delta_{ij}(\sigma)\Delta_{kl}(\sigma) = N\delta_{il}\delta_{jk}\,. \tag{2.13}$$

Note also that using (2.8) with (2.11) or using (2.9) with (2.12), one obtains

$$\frac{1}{N}\,\mathrm{tr}\,M = \frac{1}{N^2}\sum_{\sigma_i\in\mathbb{Z}_N} f(\sigma)\,. \tag{2.14}$$

We define the "star-product" of two functions $f_1(\sigma)$ and $f_2(\sigma)$ by

$$f_1(\sigma)\star f_2(\sigma) \stackrel{\mathrm{def}}{=} \frac{1}{N}\,\mathrm{tr}(M_1M_2\Delta(\sigma))\,, \tag{2.15}$$

where

$$M_\alpha = \frac{1}{N^2}\sum_{\sigma_i\in\mathbb{Z}_N} f_\alpha(\sigma)\Delta(\sigma) \qquad \alpha = 1,2\,. \tag{2.16}$$

The star-product can be written explicitly in terms of $f_\alpha(\sigma)$ as

$$f_1(\sigma)\star f_2(\sigma) = \frac{1}{N^2}\sum_{\xi_i\in\mathbb{Z}_N}\sum_{\eta_i\in\mathbb{Z}_N} f_1(\xi)f_2(\eta)\mathrm{e}^{-2\pi i\epsilon_{jk}(\xi_j-\sigma_j)(\eta_k-\sigma_k)/N}\,, \tag{2.17}$$

where ϵ_{ij} is an antisymmetric tensor with $\epsilon_{12} = 1$. This formula can be derived in the following way. Substituting (2.16) into (2.15) and using the definition (2.6) of $\Delta(\sigma)$, one obtains

$$f_1(\sigma)\star f_2(\sigma) = \left(\frac{1}{N^2}\right)^2\sum_{\xi\eta}\sum_{kpq} f_1(\xi)f_2(\eta)\mathrm{e}^{2\pi i(p\cdot\xi+q\cdot\eta+k\cdot\sigma)}\frac{1}{N}\,\mathrm{tr}(J_pJ_qJ_k)\,. \tag{2.18}$$

From the definition of J_k, one easily finds that

$$\frac{1}{N}\,\mathrm{tr}(J_pJ_qJ_k) = \mathrm{e}^{2\pi i\epsilon_{ij}p_iq_j/N}\,\delta_{k+p+q,0\ (\mathrm{mod}\ N)}\,. \tag{2.19}$$

Integration over k, p and q in (2.18) yields eq. (2.17).

In order to confirm that the star-product (2.17) is a proper discretized version of the usual star-product in the continuum, we rewrite it in terms of Fourier modes. We make a Fourier mode expansion of $f_\alpha(\sigma)$ as

$$\begin{aligned} f_\alpha(\sigma) &= \sum_{k_i\in\mathbb{Z}_N}\tilde{f}_\alpha(k)\,\mathrm{e}^{2\pi ik\cdot\sigma/N}\,, \\ \tilde{f}_\alpha(k) &= \frac{1}{N^2}\sum_{\sigma_i\in\mathbb{Z}_N} f_\alpha(\sigma)\,\mathrm{e}^{-2\pi ik\cdot\sigma/N}\,. \end{aligned} \tag{2.20}$$

Integrating (2.18) over k, ξ and η, one obtains

$$f_1(\sigma) \star f_2(\sigma) = \sum_{p_i \in \mathbb{Z}_N} \sum_{q_i \in \mathbb{Z}_N} \tilde{f}_1(p) \tilde{f}_2(q)\, \mathrm{e}^{2\pi i \epsilon_{jk} p_j q_k / N} \mathrm{e}^{2\pi i (p+q)\cdot\sigma/N}\,, \tag{2.21}$$

which can be compared to the usual definition of the star-product in the continuum

$$f_1(\sigma) \star f_2(\sigma) = f_1(\sigma) \exp\Big(i \frac{1}{2} \theta \epsilon_{ij} \overleftarrow{\partial_i} \overrightarrow{\partial_j} \Big) f_2(\sigma)\,. \tag{2.22}$$

In the present case, the noncommutativity parameter θ is of order $1/N$, and therefore the star-product reduces to the ordinary product in the large-N limit if $f_\alpha(\sigma)$ contains only lower Fourier modes ($p_j, q_j \ll \sqrt{N}$).

It is obvious from the definition (2.15) that the algebraic properties of the star-product are exactly those of the matrix product. Namely, it is associative but not commutative. Note also that due to (2.14), summing a function $f(\sigma)$ over σ corresponds to taking the trace of the corresponding matrix M. Therefore,

$$\sum_{\sigma_i \in \mathbb{Z}_N} f_1(\sigma) \star f_2(\sigma) \star \cdots \star f_n(\sigma) \tag{2.23}$$

is invariant under cyclic permutations of $f_\alpha(\sigma)$. What is not obvious solely from the algebraic properties is that

$$\sum_{\sigma_i \in \mathbb{Z}_N} f_1(\sigma) \star f_2(\sigma) = \sum_{\sigma_i \in \mathbb{Z}_N} f_1(\sigma) f_2(\sigma)\,, \tag{2.24}$$

which can be shown by using the definition (2.15) with eq. (2.9).

For later convenience, let us define the Moyal bracket by

$$\begin{aligned} \{\!\{ f_1(\sigma), f_2(\sigma) \}\!\} &\stackrel{\text{def}}{=} i \frac{N}{4\pi} \Big(f_1(\sigma) \star f_2(\sigma) - f_2(\sigma) \star f_1(\sigma) \Big) \\ &= - \frac{N}{2\pi} \sum_{pq} \tilde{f}_1(p) \tilde{f}_2(q) \sin\left(\frac{2\pi \epsilon_{jk} p_j q_k}{N} \right) \mathrm{e}^{2\pi i (p+q)\cdot\sigma/N}\,. \end{aligned} \tag{2.25}$$

We also define the Poisson bracket on a discretized worldsheet. Namely when we define the Poisson bracket

$$\{ f_1(\sigma), f_2(\sigma) \} \stackrel{\text{def}}{=} \epsilon_{ij} \frac{\partial f_1(\sigma)}{\partial \sigma_i} \frac{\partial f_2(\sigma)}{\partial \sigma_j}\,, \tag{2.26}$$

we assume that the derivatives are given by the lattice derivatives

$$\frac{\partial f(\sigma)}{\partial \sigma_i} = \frac{1}{2a} \Big(f(\sigma + \hat{i}) - f(\sigma - \hat{i}) \Big)\,, \tag{2.27}$$

where $a = 2\pi/N$ is the lattice spacing. The Poisson bracket thus defined can be written in terms of Fourier modes as

$$\begin{aligned} \{ f_1(\sigma), f_2(\sigma) \} = &- \left(\frac{N}{2\pi} \right)^2 \sum_{pq} \tilde{f}_1(p) \tilde{f}_2(q) \epsilon_{jk} \sin\left(\frac{2\pi p_j}{N} \right) \sin\left(\frac{2\pi q_k}{N} \right) \times \\ &\times \mathrm{e}^{2\pi i (p+q)\cdot\sigma/N}\,. \end{aligned} \tag{2.28}$$

Note that the appearance of sines is due to discretization of the worldsheet. The Moyal bracket (2.25) and the Poisson bracket (2.28) agree in the large-N limit if non-vanishing Fourier modes are those with $p_j, q_j \ll \sqrt{N}$.

3. Matrix model as non-commutative worldsheet theory

Let us proceed to the derivation of the equivalence (1.1) between the matrix model and the non-commutative worldsheet theory. We start from the matrix model with the action

$$S = -\frac{1}{4g^2} \operatorname{tr}\left([A_\mu, A_\nu]^2\right), \tag{3.1}$$

which can be obtained from the zero-volume limit of D-dimensional SU(N) Yang-Mills theory.[4] The indices μ run from 1 to D.

As we have done in (2.11), we write the $N \times N$ hermitean matrices A_μ in (3.1) as

$$A_\mu = \frac{1}{N^2} \sum_{\sigma_i \in \mathbb{Z}_N} X_\mu(\sigma) \Delta(\sigma), \tag{3.2}$$

where $X_\mu(\sigma)$ is a field on the discretized 2d torus obeying periodic boundary conditions. Since A_μ is hermitean, $X_\mu(\sigma)$ is real, due to the hermiticity of $\Delta(\sigma)$. Equation (3.2) can be inverted as

$$X_\mu(\sigma) = \frac{1}{N} \operatorname{tr}\left(A_\mu \, \Delta(\sigma)\right). \tag{3.3}$$

We regard σ as (discretized) worldsheet coordinates and $X_\mu(\sigma)$ as the embedding function of the worldsheet into the target space.

Using the map discussed in the previous section we can rewrite the action (3.1) in terms of $X_\mu(\sigma)$ as[5]

$$\begin{aligned} S &= -\frac{1}{4g^2 N} \sum_{\sigma_i \in \mathbb{Z}_N} \Big(X_\mu(\sigma) \star X_\nu(\sigma) - X_\nu(\sigma) \star X_\mu(\sigma) \Big)^2 \\ &= \frac{1}{g^2 N} \left(\frac{2\pi}{N}\right)^2 \sum_{\sigma_i \in \mathbb{Z}_N} \{\{X_\mu(\sigma), X_\nu(\sigma)\}\}^2 . \end{aligned} \tag{3.4}$$

If $X_\mu(\sigma)$ are sufficiently smooth functions of σ, the Moyal bracket can be replaced by the Poisson bracket, and therefore we obtain the Schild action

$$S_{\text{Schild}} = \frac{1}{g^2 N} \left(\frac{2\pi}{N}\right)^2 \sum_{\sigma_i \in \mathbb{Z}_N} \{X_\mu(\sigma), X_\nu(\sigma)\}^2 . \tag{3.5}$$

[4] For $D = 10$, the action (3.1) is just the bosonic part of the IIB matrix model [11]. The dynamical aspects of this kind of matrix models for various D with or without supersymmetry have been studied by many authors [6, 7, 18, 22] both numerically and analytically.

[5] While this work was being completed, we received a preprint [23] where the worldsheet theory (3.4) is discussed from a different point of view.

Let us discuss the symmetry of the theory (3.4), which shall be important in our analysis. For that we recall that the model (3.1) is invariant under the SU(N) transformation

$$A_\mu \longrightarrow g\, A_\mu\, g^\dagger\,. \tag{3.6}$$

From the matrix-field correspondence described in the previous section, one easily finds that the field $X_\mu(\sigma)$ defined through (3.3) transforms as

$$X_\mu(\sigma) \longrightarrow g(\sigma) \star X_\mu(\sigma) \star g(\sigma)^*\,, \tag{3.7}$$

where $g(\sigma)$ is defined by

$$g(\sigma) = \frac{1}{N}\,\mathrm{tr}\Big(g\,\Delta(\sigma)\Big)\,. \tag{3.8}$$

The fact that $g \in \mathrm{SU}(N)$ implies that $g(\sigma)$ is star-unitary;

$$g(\sigma)^* \star g(\sigma) = g(\sigma) \star g(\sigma)^* = 1\,. \tag{3.9}$$

The action (3.4) is invariant under the star-unitary transformation (3.7) as it should. We shall refer to this invariance as "gauge" degrees of freedom in what follows.

Even if $X_\mu(\sigma)$ is a smooth function of σ for a particular choice of gauge, it can be made rough by making a rough star-unitary transformation. Let us quote an analogous situation in lattice gauge theory. In the weak coupling limit, the configurations can be made very smooth by a proper choice of the gauge. However, without gauge fixing, they are as rough as could be due to the unconstrained gauge degrees of freedom. Similarly when we discuss the smoothness of $X_\mu(\sigma)$, we should subtract the roughness due to the gauge degrees of freedom appropriately. Therefore, a natural question one should ask is whether there exists at all a gauge choice that makes $X_\mu(\sigma)$ relatively smooth functions of σ.

In order to address this question, we specify a gauge-fixing condition by first defining the roughness of the worldsheet configuration $X_\mu(\sigma)$ and then choosing a gauge so that the roughness is minimized. A natural definition of roughness is

$$I = \frac{1}{2N^2} \sum_{\sigma_i \in \mathbb{Z}_N} \sum_{j=1}^{2} \Big(X_\mu(\sigma + \hat{j}) - X_\mu(\sigma)\Big)^2\,, \tag{3.10}$$

which is Lorentz invariant. The gauge fixing is analogous to the Landau gauge in gauge theories. The roughness functional (3.10) can be written conveniently in terms of A_μ as (See appendix A for the derivation.)

$$I = \frac{1}{2N} \sum_{IJ} \left[4\sin^2 \frac{\pi(I-J)}{N} |(A_\mu)_{IJ}|^2 + \left|(A_\mu)_{IJ} - (A_\mu)_{I+\frac{N-1}{2}, J+\frac{N-1}{2}}\right|^2 \right]. \tag{3.11}$$

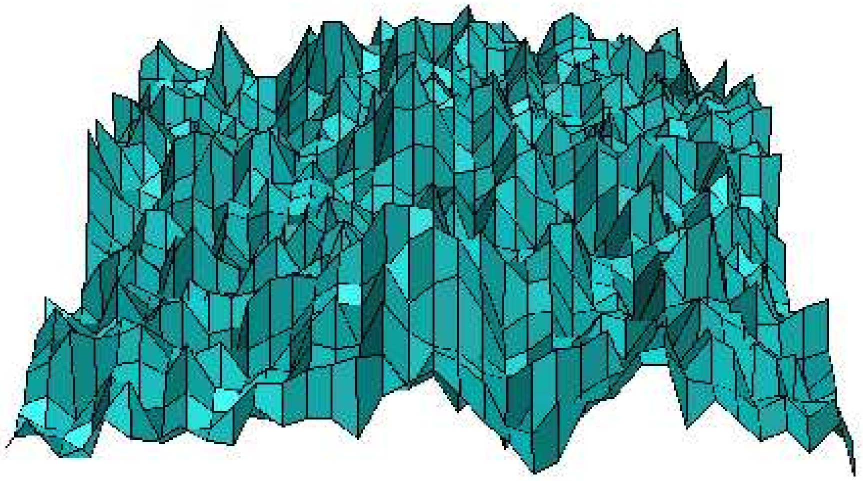

Figure 1: A typical $N = 35$ $X_\mu(\sigma)$ ($\mu = 1$) configuration after the gauge fixing.

4. Results

Our numerical calculation starts with generating configurations of the model (3.1) for $D = 4$ and $N = 15, 25, 35$ using the method described in ref. [7]. For each configuration we minimize the roughness functional I defined by eq. (3.11) with respect to the SU(N) transformation (3.6). We perform 2000 sweeps per configuration, where a sweep is the minimization of I with respect to all SU(2) subgroups of SU(N) [7, 18]. From the configuration A_μ obtained after the SU(N) transformation that minimizes I, we calculate through (3.3) the worldsheet configuration $X_\mu(\sigma)$. When we define an ensemble average $\langle \cdot \rangle$ in what follows, we assume that it is taken with respect to $X_\mu(\sigma)$ *after* the gauge fixing. The number of configurations used for an ensemble average is 658, 100 and 320 for $N = 15, 25, 35$, respectively. We note that in the present model, finite N effects is known [7] to appear as a $1/N^2$ expansion. Also, the large-N factorization is clearly observed for $N = 16, 32$ [7]. (We have checked that it occurs for $N = 15, 25, 35$ as well.) We therefore consider that the N we use in the present work is sufficiently large to discuss the large-N asymptotics.

We also note that the parameter g, which appears in the action (3.4), can be absorbed by the field redefinition $X'_\mu(\sigma) = \frac{1}{\sqrt{g}} X_\mu(\sigma)$. Therefore, g is merely a scale parameter, and one can determine the g dependence of all the observables on dimensional grounds. The results will be stated in such a way that they do not depend on the choice of g.

In figure 1 we show a typical worldsheet configuration for $N = 35$ after the gauge fixing. We observe that the worldsheet has no spikes. We compute the Fourier modes

$\tilde{X}_\mu(k)$ of $X_\mu(\sigma)$ through

$$\tilde{X}_\mu(k) \stackrel{\text{def}}{=} \frac{1}{N^2} \sum_{\sigma_i \in \mathbb{Z}_N} X_\mu(\sigma) \mathrm{e}^{-2\pi i k \cdot \sigma / N} = \frac{1}{N} \operatorname{tr}(A_\mu J_k)\,, \tag{4.1}$$

where the range of k_1 and k_2 are chosen to be $-(N-1)/2$ to $(N-1)/2$. Figure 2 describes how the same worldsheet configuration shown in figure 1 looks like if we cut off[6] the Fourier modes higher than k_c. We find that the configuration obtained by keeping only a few lower Fourier modes already captures the characteristic behavior of the original configuration. We have checked that this is *not* the case if we do not fix the gauge.

We measure the fluctuation of the surface, which is given by[7]

$$R^2 \stackrel{\text{def}}{=} \frac{1}{N^4} \sum_{\sigma_i, \sigma_i' \in \mathbb{Z}_N} \left\{ X_\mu(\sigma) - X_\mu(\sigma') \right\}^2 = \frac{2}{N^2} \sum_{\sigma_i \in \mathbb{Z}_N} X_\mu(\sigma)^2 = \frac{2}{N} \operatorname{tr}(A_\mu^2)\,. \tag{4.2}$$

The average of the fluctuation is finite for finite N, and it is plotted in figure 3. The power-law fit to the large-N behavior $\langle R^2 \rangle \sim g N^p$ yields $p = 0.493(3)$, which is consistent with the result $p = 1/2$ obtained in ref. [7]. Although the finiteness of the fluctuation already implies a certain stability of the worldsheet, we note that the fluctuation defined by the l.h.s. of (4.2) is actually invariant under the star-unitary transformation (3.7). In particular, the fluctuation is finite even before the gauge fixing. Therefore, the smoothness of $X_\mu(\sigma)$ (which appears only after the gauge fixing) is a notion which is stronger than the finiteness of the fluctuation.

Let us point out also that the roughness functional I actually represents the average link length in the target space. We therefore plot $\langle I \rangle$ in figure 3 and compare it with the fluctuation (4.2). The former is observed to be smaller than the latter[8] which is consistent with the observed smoothness of $X_\mu(\sigma)$. The ratio $\langle I \rangle / \langle R^2 \rangle$ is $0.364(1)$, $0.338(2)$, $0.3290(5)$ for $N = 15, 25, 35$, respectively, which may be fitted to a power-law behavior $\langle I \rangle / \langle R^2 \rangle \sim N^{-0.120(4)}$. This indicates a tendency that the worldsheet is getting smoother as N increases.

In order to quantify the smoothness of $X_\mu(\sigma)$ further, let us examine the Fourier mode amplitudes. We first note that there is a relation

$$\sum_{k_i \in \mathbb{Z}_N} \left\langle \tilde{X}_\mu(k) \tilde{X}_\mu(-k) \right\rangle = \left\langle \frac{1}{N^2} \sum_{\sigma_i \in \mathbb{Z}_N} X_\mu(\sigma)^2 \right\rangle \sim \mathrm{O}(g N^{1/2})\,. \tag{4.3}$$

[6] More precisely, we keep the modes $\tilde{X}_\mu(k)$ with $k_1 \le k_c$ and $k_2 \le k_c$ and set the other modes to zero by hand.

[7] We note that the action (3.1) is invariant under $A_\mu \mapsto A_\mu + \alpha_\mu \mathbb{1}_N$. Therefore, the trace part of A_μ is completely decoupled from the dynamics. We fix these degrees of freedom by imposing the matrices A_μ to be traceless. In the language of the worldsheet theory (3.4), the symmetry corresponds to the translational invariance $X_\mu(\sigma) \mapsto X_\mu(\sigma) + \text{const.}$. The tracelessness condition for A_μ maps to $\sum_{\sigma_i \in \mathbb{Z}_N} X_\mu(\sigma) = 0$.

[8] If we do not fix the gauge, we observe that the two quantities $\langle R^2 \rangle$ and $\langle I \rangle$ coincide, meaning that the $X_\mu(\sigma)$ is completely rough.

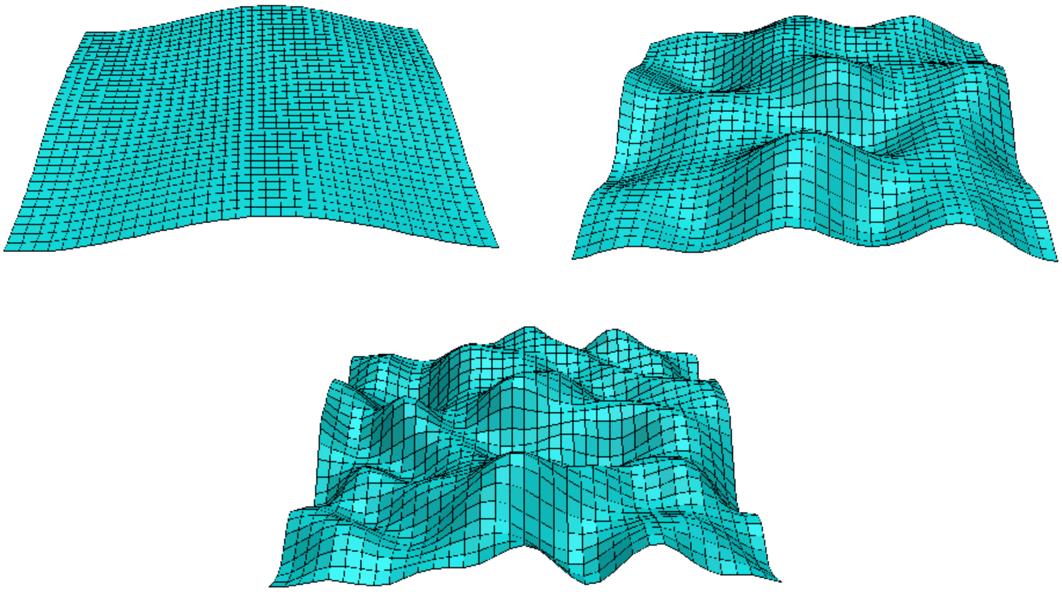

Figure 2: The effect of cutting off Fourier modes higher than k_c on the worldsheet of figure 1. On the left $k_c = 1$, on the right $k_c = 3$ and on the bottom $k_c = 5$.

This motivates us to plot $\langle \tilde{X}_\mu(k)\tilde{X}_\mu(-k)\rangle/(gN^{1/2})$ with this particular normalization. The results are shown in figure 4 and figure 5. We find a good scaling in N; data points for different N fall on top of each other. The discrepancy in the large-k region can be understood as a finite N effect. The k dependence of the Fourier mode amplitudes suggests that the higher modes are indeed suppressed. Moreover we find that there exist a power-law behavior[9]

$$\frac{1}{gN^{1/2}} \left\langle \tilde{X}_\mu(k)\tilde{X}_\mu(-k) \right\rangle \sim \text{const.} \cdot |k|^{-q} , \tag{4.4}$$

where $|k| = \sqrt{(k_1)^2 + (k_2)^2}$. Assuming that the constant coefficient on the r.h.s. of (4.4) is independent of N, as suggested by the observed scaling in N, the power q must be $q > 2$ in order that the sum on the l.h.s. of (4.3) may be convergent in the large-N limit. The power q extracted from $N = 35$ data is $q = 1.96(5)$, which may imply that we have not reached sufficiently large k (due to the finite N effect mentioned above) to extract the correct power. Although we have seen that the amplitudes of the higher Fourier modes are suppressed, we should remember that their number for fixed $|k|$ grows linearly with $|k|$. Therefore, the higher modes can still be non negligible.

[9] If we do not fix the gauge, we observe that the r.h.s. of (4.4) is replaced by a constant independent of k, which means that the worldsheet behaves like a white noise. The constant is proportional to $1/N^2$, as expected from the relation (4.3), which is gauge independent. Note, in particular, that the scaling behavior observed in figure 5 emerges only after the gauge fixing.

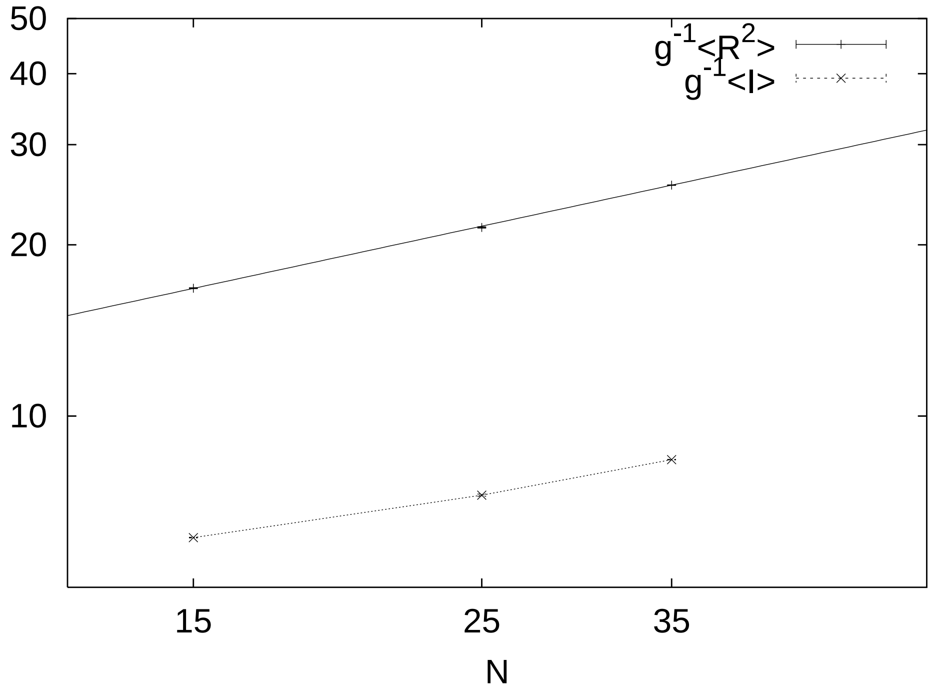

Figure 3: The fluctuation of the surface R^2 and the roughness functional I, which represents the average link length in the target space, are plotted against N with the normalization factor g^{-1}. The solid straight line is a fit to the power-law behavior $\langle R^2 \rangle \propto g N^{0.493(3)}$. The dotted line is drawn to guide the eye.

Let us turn to the question whether the action S in (3.4) approaches the Schild action $S_{\rm Schild}$ in (3.5) in the large-N limit. In terms of Fourier modes, the two actions read

$$\begin{aligned} S &= -\frac{N}{g^2} \sum_{klp} \tilde{X}_\mu(k) \tilde{X}_\nu(l) \tilde{X}_\mu(p) \tilde{X}_\nu(-k-l-p) \times \\ &\qquad \times \sin\left(\frac{2\pi}{N} \epsilon_{nm} k_n l_m\right) \sin\left(\frac{2\pi}{N} \epsilon_{rs} p_r (k+l)_s\right), \\ S_{\rm Schild} &= -\frac{N}{g^2} \left(\frac{N}{2\pi}\right)^2 \sum_{klp} \tilde{X}_\mu(k) \tilde{X}_\nu(l) \tilde{X}_\mu(p) \tilde{X}_\nu(-k-l-p) \epsilon_{ij} \sin\left(\frac{2\pi}{N} k_i\right) \times \\ &\qquad \times \sin\left(\frac{2\pi}{N} l_j\right) \epsilon_{rs} \sin\left(\frac{2\pi}{N} p_r\right) \times \\ &\qquad \times \sin\left(\frac{2\pi}{N} (k+l+p)_s\right). \end{aligned} \tag{4.5}$$

If the higher Fourier modes can be neglected one can see from eqs. (4.5) that $S = S_{\rm Schild}$ in the large-N limit. We measure both quantities and plot the results in figure 6. The average of the action (3.4) is known [7] analytically $\langle S \rangle = N^2 - 1$, which

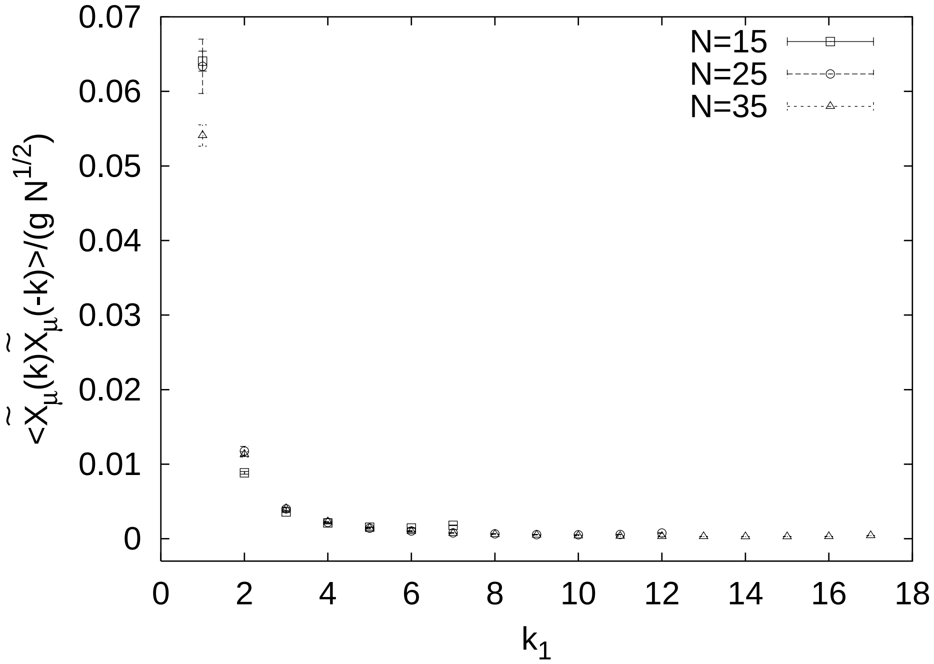

Figure 4: The Fourier mode amplitudes $\langle \tilde{X}_\mu(k)\tilde{X}_\mu(-k)\rangle/(gN^{1/2})$ are plotted against k_1 ($k_2 = k_1$) for $N = 15, 25, 35$.

is clearly reproduced from our data. On the other hand, $\langle S_{\text{Schild}}\rangle$ is much larger, and moreover it grows much faster, the growth being close to $\mathrm{O}(N^4)$. Therefore we can safely conclude that the two quantities do not coincide in the large-N limit.

The disagreement of the two actions (3.4) and (3.5) in the large-N limit implies that the higher modes play a crucial role. In order to see their effects explicitly, we cut off the Fourier modes higher than k_c. In figure 7, we plot the average of the actions thus calculated against k_c for $N = 35$. The two actions with the cutoff at k_c are indeed identical for small k_c, but they start to deviate from each other as k_c increases, ending up with totally different values at $k_c = (N-1)/2$, i.e. when all the modes are included.

Finally, let us discuss the extrinsic geometry of the worldsheet. One may define the Hausdorff dimension d_H of the worldsheet through

$$\frac{\langle A\rangle}{\ell^2} \propto \left(\frac{\sqrt{\langle R^2\rangle}}{\ell}\right)^{d_H} \qquad \text{as} \qquad N \to \infty\,, \tag{4.6}$$

where R, which is defined by eq. (4.2), represents the extent of the worldsheet in the target space. We have defined A, the total area of the worldsheet in the target space, by

$$A = a^2 \sum_{\sigma_i \in \mathbb{Z}_N} \sqrt{\{X_\mu(\sigma), X_\nu(\sigma)\}^2}\,, \tag{4.7}$$

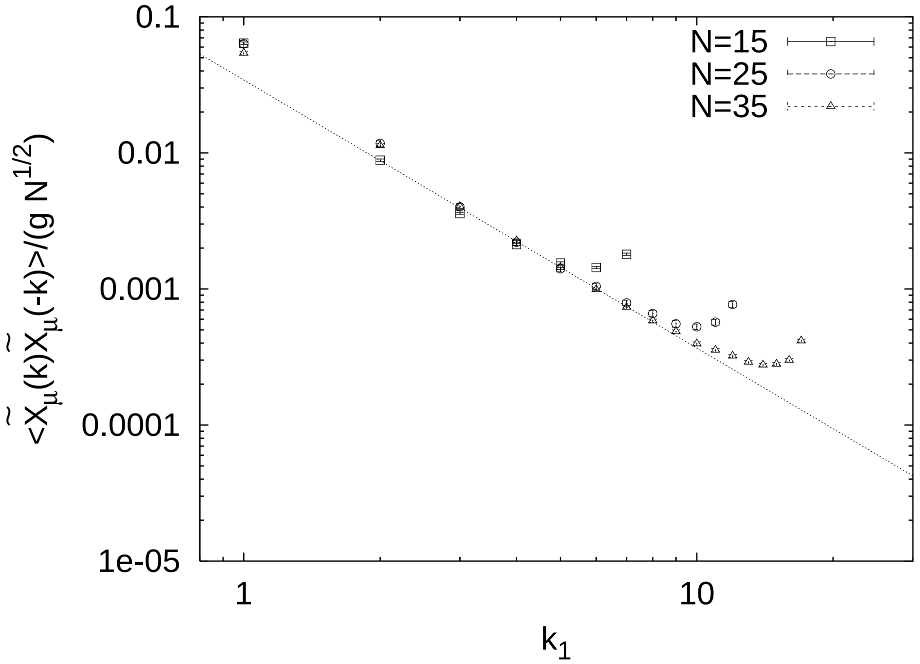

Figure 5: The Fourier mode amplitudes $\langle \tilde{X}_\mu(k)\tilde{X}_\mu(-k)\rangle/(gN^{1/2})$ in figure 4 are now plotted in the log-log scale in order to visualize the power-law behavior. The straight line is a fit to $C|k|^{-q}$ with $q = 1.96(5)$ for the $N = 35$ data.

where the Poisson bracket $\{\cdot\}$ is defined by (2.26) and $a = 2\pi/N$ is the lattice spacing on the worldsheet. Equation (4.7) is nothing but the Nambu-Goto action. The scale parameter ℓ in eq. (4.6) should be introduced in order to make the equation consistent dimensionally. A natural choice of the fundamental scale ℓ is the average link length in the target space, which is represented by $\sqrt{\langle I\rangle}$, where I is the roughness functional defined by eq. (3.11). We observe, as expected, that $\langle A\rangle \sim N^2\langle I\rangle$, which means that the l.h.s. of (4.6) is of $\mathrm{O}(N^2)$. As we have seen in figure 3, we observe that $\langle I\rangle/\langle R^2\rangle \sim N^{-0.120(4)}$. This means that the Hausdorff dimension d_H defined by (4.6) with the choice $\ell = \sqrt{\langle I\rangle}$ is $d_H \sim 33$, which might suggest that actually $d_H = \infty$. Therefore, the extrinsic geometry of the embedded worldsheet is described by a crumpled surface.

5. Conclusions

In this paper we have studied the star-Schild action (1.2) resulting from the zero-volume limit of $\mathrm{SU}(N)$ Yang-Mills theory. We find that the star-Schild action does not approach the Schild action (1.4) due to the important role played by the ever-increasing number of higher modes. This has some implications to the ideas presented long ago [9] that it might be that the Schild action represents the large-N QCD

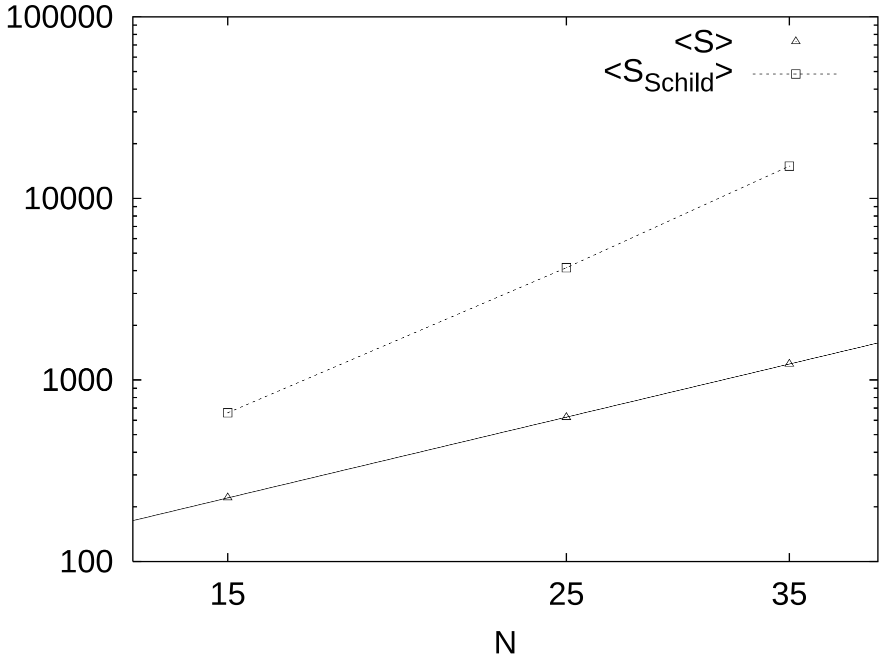

Figure 6: Plot of $\langle S\rangle$ and $\langle S_{\rm Schild}\rangle$ as a function of N. The solid line represents the exact result $\langle S\rangle = (N^2-1)$. The dotted line is drawn to guide the eye.

action. From our results the two actions differ more and more with increasing N. The Poisson bracket increases much faster with N than the corresponding Moyal bracket. Our conclusion is therefore that QCD strings would be described by a non-commutative string theory defined by the star-Schild action, rather than the standard Schild action.

As we have seen, it is possible to find a star-unitary transformation $g(\sigma)$ such that the surfaces defined by the star-Schild action are regular (i.e., do not have long spikes), and this action therefore defines a new type of string theory. The theory is invariant under star-unitary transformations, which generalize the area-preserving diffeomorphisms, the invariance of the usual Schild action. As we discuss in the appendix B, the reparametrization invariance of the worldsheet fields $X_\mu(\sigma)$ is restricted to linear transformations of the σ's in the case of the star-Schild action. (Note, however, that the reparametrization invariance of the usual Schild action is also much reduced relative to the Nambu-Goto action.) The star-unitary transformations transform the worldsheet configuration in such a way that the changes cannot be absorbed by a reparametrization. In obtaining a regular surface we have chosen a particular "gauge". The regularity will not be changed drastically under smooth star-unitary transformations. However, if we do not fix the gauge, we obtain spiky surfaces, which is connected to a regular surface by a rough star-unitary transformation $g(\sigma)$. The main point is that it is at all *possible* to obtain a regular surface by fixing the gauge properly.

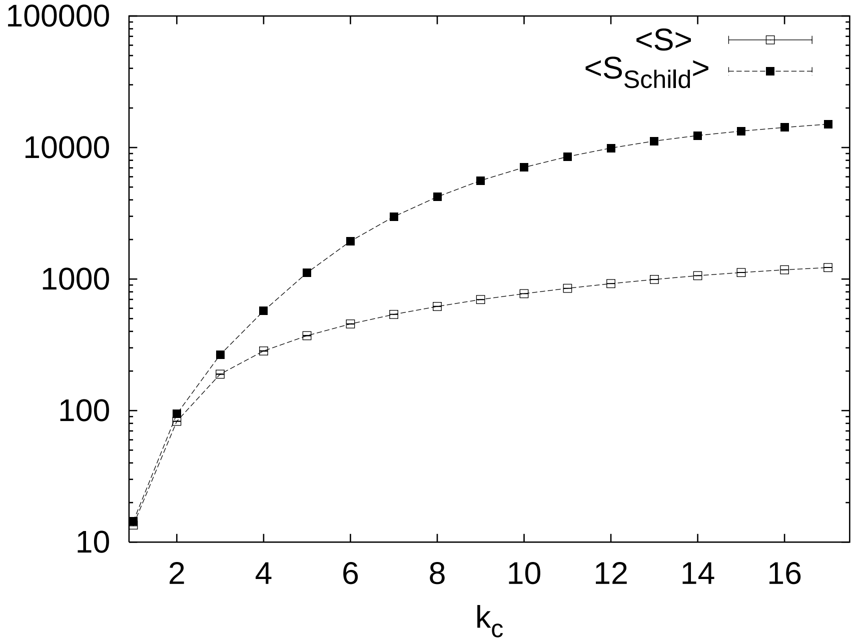

Figure 7: The two actions $\langle S_{\rm Schild} \rangle$ and $\langle S \rangle$ are computed by cutting off the Fourier modes higher than $k > k_{\rm c}$ by hand. The results are plotted as a function of the mode cutoff $k_{\rm c}$ for $N = 35$.

A possible intuitive understanding of the regularity of the worldsheet in the non-commutative string is that the action contains higher derivatives in the star-product. Let us recall that one of the motivations for introducing extrinsic curvature (which also contains higher derivatives in a different combination) was [4] that this extra term in the action makes the worldsheet more stiff. One would also expect a similar effect from the introduction of higher derivatives, because an extremely rough worldsheet with long spikes would have at least some derivatives rather large. Although the star-product contains these derivatives in a special combination, it is difficult for all these large derivatives to cancel, and hence a surface dominated by long spikes would not be preferred. This is confirmed by the observation that the average link length is much smaller than the average extent. The extrinsic geometry of the embedded surface, on the other hand, is described by a crumpled surface with a large Hausdorff dimension.

It would be of interest to address the issues studied in this paper in the supersymmetric case using the numerical method developed in ref. [18]. We hope to report on it in a future publication.

Acknowledgments

We thank Jan Ambjørn for interesting discussions on instabilities in bosonic strings. We are also grateful to Z. Burda for helpful comments. K.N.A. acknowledges interesting discussions with E. Floratos, E. Kiritsis and G. Savvidy. K.N.A.'s research was partially supported by the RTN grants HPRNCT-1999-00161, HPRN-CT-2000-00122 and HPRN-CT-2000-00131 and a National Fellowship Foundation of Greece (IKY) postdoctoral fellowship.

A. Derivation of the roughness functional

In this appendix, we rewrite the roughness functional (3.10) in terms of the matrices A_μ and derive (3.11). For that purpose, we introduce $N \times N$ unitary matrices

$$\begin{aligned} D_1 &= (\Gamma_2^\dagger)^{\frac{N-1}{2}} \, , \\ D_2 &= (\Gamma_1)^{\frac{N-1}{2}} \, , \end{aligned} \tag{A.1}$$

which satisfy

$$D_j \Gamma_i D_j^\dagger = \mathrm{e}^{-2\pi i \delta_{ij}/N} \Gamma_i \, . \tag{A.2}$$

One can check that

$$D_j \Delta(\sigma) D_j^\dagger = \Delta(\sigma - \hat{j}) \, , \tag{A.3}$$

which implies

$$X_\mu(\sigma + \hat{j}) = \frac{1}{N} \, \mathrm{tr} \Big(D_j A_\mu D_j^\dagger \Delta(\sigma) \Big) \, . \tag{A.4}$$

Thus the matrix D_j plays the role of a shift operator in the j-direction. Now we can rewrite the roughness functional I in terms of the matrices A_μ as

$$\begin{aligned} I &= \frac{1}{2N} \sum_j \mathrm{tr}(D_j A_\mu D_j^\dagger - A_\mu)^2 \\ &= \frac{1}{2N} \sum_{IJ} \sum_j \Big| (D_j A_\mu D_j^\dagger)_{IJ} - (A_\mu)_{IJ} \Big|^2 \, . \end{aligned} \tag{A.5}$$

Using the explicit form of Γ_μ given by eq. (2.2), we obtain

$$\begin{aligned} (D_1 A_\mu D_1^\dagger)_{IJ} &= (A_\mu)_{IJ} \mathrm{e}^{2\pi i (I-J)/N} \, , \\ (D_2 A_\mu D_2^\dagger)_{IJ} &= (A_\mu)_{I+\frac{N-1}{2}, J+\frac{N-1}{2}} \, , \end{aligned} \tag{A.6}$$

which yields (3.11).

B. Star-unitary invariance and area-preserving diffeomorphisms

In this appendix, we discuss the relationship between the symmetry of the Schild action and that of the star-Schild action. Here only we consider that the worldsheet is given by an infinite two-dimensional flat space parametrized by the continuous variables σ_1 and σ_2. Let us define the Schild and star-Schild actions

$$\begin{aligned} I_1 &= \int \mathrm{d}^2\sigma \, \{\phi_1(\sigma), \phi_2(\sigma)\}^2 \,, \\ I_2 &= \int \mathrm{d}^2\sigma \, \{\{\phi_1(\sigma), \phi_2(\sigma)\}\}_\theta^2 \,, \end{aligned} \tag{B.1}$$

where the Poisson and Moyal brackets are defined by

$$\begin{aligned} \{\phi_1(\sigma), \phi_2(\sigma)\} &= \epsilon_{ij} \frac{\partial \phi_1}{\partial \sigma_i} \frac{\partial \phi_2}{\partial \sigma_j} \,, \\ \{\{\phi_1(\sigma), \phi_2(\sigma)\}\}_\theta &= \frac{1}{\theta} \phi_1(\sigma) \sin(\theta \epsilon_{ij} \overleftarrow{\partial_i} \overrightarrow{\partial_j}) \phi_2(\sigma) \,. \end{aligned} \tag{B.2}$$

The Schild action I_1 is invariant under the area-preserving diffeomorphism

$$\sigma_i \longmapsto \sigma_i + \epsilon_{ij} \partial_j f(\sigma) \,, \tag{B.3}$$

where $f(\sigma)$ is some infinitesimal real function of σ. Under the infinitesimal area-preserving diffeomorphism, the fields transform as a scalar $\phi_\alpha(\sigma) = \phi'_\alpha(\sigma')$, so that one can state the invariance as the one under the field transformation

$$\phi_\alpha(\sigma) \longmapsto \phi_\alpha(\sigma) + \{f(\sigma), \phi_\alpha(\sigma)\} \,. \tag{B.4}$$

On the other hand, the star-Schild action I_2 is invariant under the star-unitary transformation

$$\phi_\alpha(\sigma) \longmapsto \phi_\alpha(\sigma) + \{\{f(\sigma), \phi_\alpha(\sigma)\}\}_\theta \,. \tag{B.5}$$

Obviously, this transformation (B.5) reduces to (B.4) if $\phi_\alpha(\sigma)$ and $f(\sigma)$ do not contain higher Fourier modes compared with $\theta^{-1/2}$.

In general the two transformations (B.4) and (B.5) differ. However, we note that they become identical if $f(\sigma)$ contains terms only up to quadratic in σ as

$$f(\sigma) = a_i \sigma_i + b_{ij} \sigma_i \sigma_j \,, \tag{B.6}$$

where b_{ij} is a real symmetric tensor. From (B.3), one finds that the corresponding coordinate transformation is

$$\sigma_i \longmapsto \sigma_i + (v_i + \lambda_{ij} \sigma_j) \,, \tag{B.7}$$

where $v_i = \epsilon_{il} a_l$ and $\lambda_{ij} = \epsilon_{il} b_{lj}$, which is traceless. This transformation includes the euclidean group, namely translation and rotation. Thus we find that the linear (finite) transformation of the coordinates $\sigma'_i = \Lambda_{ij} \sigma_j + v_i$, where $\det \Lambda = 1$, can be expressed as a star-unitary transformation. In other words, the reparametrization invariance of the star-Schild action is restricted to such linear transformations.

References

[1] O. Alvarez, *The static potential in string models*, *Phys. Rev.* **D 24** (1981) 440;
M.E. Cates, *The Liouville field theory of random surfaces: when is the bosonic string a branched polymer?*, *Europhys. Lett.* **8** (1988) 656.

[2] J. Ambjørn, B. Durhuus and J. Frohlich, *Diseases of triangulated random surface models and possible cures*, *Nucl. Phys.* **B 257** (1985) 433;
J. Ambjørn, B. Durhuus, J. Frohlich and P. Orland, *The appearance of critical dimensions in regulated string theories*, *Nucl. Phys.* **B 270** (1986) 457;
For a recent review, see J. Ambjørn, B. Durhuus and T. Jonsson, *Quantum geometry*, Cambridge Monographs, 1997.

[3] S. Oda and T. Yukawa, *Type-IIB random superstrings*, *Prog. Theor. Phys.* **102** (1999) 215 [hep-th/9903216];
P. Bialas, Z. Burda, B. Petersson and J. Tabaczek, *Large-N limit of the IKKT matrix model*, *Nucl. Phys.* **B 592** (2001) 391 [hep-lat/0007013].

[4] W. Helfrich, *Effect of thermal modulation on the rigidity of fluid membranes and interfaces*, *J. Phys.* 46 (1985) 1263;
L. Peliti and S. Leibler, *Effects of thermal fluctuations on systems with small surface tension*, *Phys. Rev. Lett.* **54** (1985) 1690;
D. Förster, *On the scale dependence, due to thermal fluctuations, of the elastic properties of membranes*, *Phys. Lett.* **A 114** (1986) 115;
A.M. Polyakov, *Fine structure of strings*, *Nucl. Phys.* **B 268** (1986) 406.

[5] J. Ambjørn, A. Irbäck, J. Jurkiewicz and B. Petersson, *The theory of dynamical random surfaces with extrinsic curvature*, *Nucl. Phys.* **B 393** (1993) 571 [hep-lat/9207008];
K. Anagnostopoulos, M. Bowick, P. Coddington, M. Falcioni, L. Han, G. Harris and E. Marinari, *Fluid random surfaces with extrinsic curvature, 2*, *Phys. Lett.* **B 317** (1993) 102 [hep-th/9308091];
J. Ambjørn, Z. Burda, J. Jurkiewicz and B. Petersson, *A random surface theory with nontrivial* γ_{string}, *Phys. Lett.* **B 341** (1995) 286 [hep-th/9408118].

[6] W. Krauth and M. Staudacher, *Finite Yang-Mills integrals*, *Phys. Lett.* **B 435** (1998) 350 [hep-th/9804199].

[7] T. Hotta, J. Nishimura and A. Tsuchiya, *Dynamical aspects of large-N reduced models*, *Nucl. Phys.* **B 545** (1999) 543 [hep-th/9811220].

[8] P. Austing and J.F. Wheater, *The convergence of Yang-Mills integrals*, *J. High Energy Phys.* **02** (2001) 028 [hep-th/0101071].

[9] E.G. Floratos and J. Iliopoulos, *A note on the classical symmetries of the closed bosonic membranes*, *Phys. Lett.* **B 201** (1988) 237;

E.G. Floratos, J. Iliopoulos, and G. Tiktopoulos, *A note on* $\mathrm{SU}(\infty)$ *classical Yang-Mills theory*, *Phys. Lett.* **B 217** (1989) 285;
D.B. Fairlie, P. Fletcher and C.K. Zachos, *Trigonometric structure constants for new infinite algebras*, *Phys. Lett.* **B 218** (1989) 203;
D.B. Fairlie and C.K. Zachos, *Infinite dimensional algebras, sine brackets and* $\mathrm{SU}(\infty)$, *Phys. Lett.* **B 224** (1989) 101;
D. Fairlie, P. Fletcher and C. Zachos, *Infinite dimensional algebras and a trigonometric basis for the classical Lie algebras*, *J. Math. Phys.* **31** (1990) 1088;
I. Bars, *Strings from reduced large-N gauge theory via area preserving diffeomorphisms*, *Phys. Lett.* **B 245** (1990) 35;
E.G. Floratos and G.K. Leontaris, *The matrix model and the non-commutative geometry of the supermembrane*, *Phys. Lett.* **B 464** (1999) 30 [hep-th/9908106].

[10] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M-theory as a matrix model: a conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [hep-th/9610043].

[11] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, *A large-N reduced model as superstring*, *Nucl. Phys.* **B 498** (1997) 467 [hep-th/9612115].

[12] T. Eguchi and R. Nakayama, *Simplification of quenching procedure for large N spin model*, *Phys. Lett.* **B 122** (1983) 59;
S.J. Rankin, $\mathrm{SU}(\infty)$ *and the large-N limit*, *Ann. Phys. (NY)* **218** (1992) 14;
A. Kavalov and B. Sakita, W_∞ *and* w_∞ *gauge theories and contraction*, *Ann. Phys. (NY)* **255** (1997) 1 [hep-th/9603024].

[13] T. Eguchi and H. Kawai, *Reduction of dynamical degrees of freedom in the large-N gauge theory*, *Phys. Rev. Lett.* **48** (1982) 1063.

[14] D.J. Gross and Y. Kitazawa, *A quenched momentum prescription for large-N theories*, *Nucl. Phys.* **B 206** (1982) 440.

[15] A. González-Arroyo and C.P. Korthals Altes, *Reduced model for large-N continuum field theories*, *Phys. Lett.* **B 131** (1983) 396.

[16] G. Bhanot, U.M. Heller and H. Neuberger, *The quenched Eguchi-Kawai model*, *Phys. Lett.* **B 113** (1982) 47.

[17] A. González-Arroyo and M. Okawa, *The twisted Eguchi-Kawai model: a reduced model for large-N lattice gauge theory*, *Phys. Rev.* **D 27** (1983) 2397.

[18] J. Ambjørn, K.N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, *Large-N dynamics of dimensionally reduced 4d* $\mathrm{SU}(N)$ *super Yang-Mills theory*, *J. High Energy Phys.* **07** (2000) 013 [hep-th/0003208].

[19] G. 't Hooft, *Monopoles, instantons and confinement*, hep-th/0010225.

[20] I. Bars and D. Minic, *Non-commutative geometry on a discrete periodic lattice and gauge theory*, *Phys. Rev.* **D 62** (2000) 105018 [hep-th/9910091].

[21] J. Ambjørn, Y.M. Makeenko, J. Nishimura and R.J. Szabo, *Finite N matrix models of noncommutative gauge theory*, *J. High Energy Phys.* **11** (1999) 029 [hep-th/9911041]; *Nonperturbative dynamics of noncommutative gauge theory*, *Phys. Lett.* **B 480** (2000) 399 [hep-th/0002158]; *Lattice gauge fields and discrete noncommutative Yang-Mills theory*, *J. High Energy Phys.* **05** (2000) 023 [hep-th/0004147].

[22] J. Nishimura, *On existence of nontrivial fixed points in large-N gauge theory in more than four dimensions*, *Mod. Phys. Lett.* **A 11** (1996) 3049 [hep-lat/9608119];
T. Suyama and A. Tsuchiya, *Exact results in $N_c = 2$ IIB matrix model*, *Prog. Theor. Phys.* **99** (1998) 321 [hep-th/9711073];
T. Nakajima and J. Nishimura, *Numerical study of the double scaling limit in two-dimensional large-N reduced model*, *Nucl. Phys.* **B 528** (1998) 355 [hep-th/9802082];
W. Krauth, H. Nicolai and M. Staudacher, *Monte Carlo approach to M-theory*, *Phys. Lett.* **B 431** (1998) 31 [hep-th/9803117];
W. Krauth and M. Staudacher, *Eigenvalue distributions in Yang-Mills integrals*, *Phys. Lett.* **B 453** (1999) 253 [hep-th/9902113];
J. Ambjørn, K.N. Anagnostopoulos, W. Bietenholz, T. Hotta and J. Nishimura, *Monte carlo studies of the IIB matrix model at large-N*, *J. High Energy Phys.* **07** (2000) 011 [hep-th/0005147];
J. Nishimura and G. Vernizzi, *Spontaneous breakdown of Lorentz invariance in IIB matrix model*, *J. High Energy Phys.* **04** (2000) 015 [hep-th/0003223]; *Brane world from IIB matrices*, *Phys. Rev. Lett.* **85** (2000) 4664 [hep-th/0007022];
S. Horata, H.S. Egawa and T. Yukawa, *Numerical analysis of the double scaling limit in the IIB matrix model*, hep-th/0005157;
S. Oda and F. Sugino, *Gaussian and mean field approximations for reduced Yang-Mills integrals*, *J. High Energy Phys.* **03** (2001) 026 [hep-th/0011175];
Z. Burda, B. Petersson and J. Tabaczek, *Geometry of reduced supersymmetric 4d Yang-Mills integrals*, hep-lat/0012001.

[23] N. Kitsunezaki and S. Uehara, *Large-N structure of IIB matrix model*, *Prog. Theor. Phys.* **105** (2001) 515 [hep-th/0010038].